

Matrix Methods in Paraxial Optics:

Introduction:

①

Consider a ray of light PQ incident on a refracting surface SQS' , which separates two media having refractive index n_1 and n_2 . $NN' \rightarrow$ be the normal to the surface at the point of incidence. The direction of the refracted ray is determined by the following two conditions or so called laws of refraction:

- i) The incident ray, the refracted ray and normal at the point of incidence all lie in same plane.
- ii) If θ_1 and θ_2 represent the angle of incidence and angle of refraction respectively; then

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{n_2}{n_1} \quad (\text{Snell's law}).$$

In case of optical systems, which consist of large no. of refracting surfaces, the above two conditions may be used to trace any ray through the system (for example in case of combination of lenses).

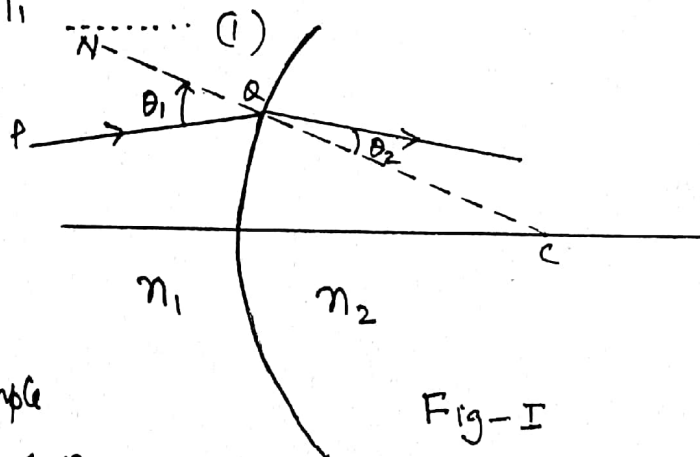


Fig-I (Refraction at a spherical surface)

For obtaining the final image in such systems; one need to make a step by step calculation of the image due to each surface, where this image act as an object for the next surface. Such calculation are complicated with increase in the elements of the optical system.

In order to deal such problems with greater ease; matrix method is applied. This method is also directly used in computers for tracing rays in complicated optical systems.

Fundamental matrix algebra:

A matrix consists of elements arranged in rows and columns, $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$.
is a 2×3 matrix having 2 rows and 3 columns (in general $m \times n$ matrix).
Matrix multiplication: Two matrices can be multiplied when the no. of columns of 1st is equal to the no. of rows of the second matrix.
ex: a $(m \times n)$ matrix can be multiplied with a $(n \times p)$ matrix.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_{(2 \times 3)}, B = \begin{pmatrix} g \\ h \\ i \end{pmatrix}_{(3 \times 1)} \text{ then } AB = \begin{pmatrix} ag + bh + ci \\ dg + eh + fi \end{pmatrix}_{(2 \times 1)} \quad \text{--- (2)}$$

How ever the product BA is meaningless.

Use of matrix for solving linear equation:

Consider two linear equations

$$\begin{aligned} x_1 &= ay_1 + by_2 \\ x_2 &= cy_1 + dy_2 \end{aligned} \quad \left. \begin{array}{l} \text{These equations can be expressed} \\ \text{as} \end{array} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{--- (3)}$$

$$\text{Further suppose } \begin{aligned} y_1 &= ez_1 + fz_2 \\ y_2 &= gz_1 + hz_2 \end{aligned} \quad \text{then } \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{--- (4)}$$

Combining (3) and (4)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{--- (5)}$$

$$\Rightarrow X = BZ \quad \text{where } B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + gh \end{pmatrix} \quad \text{--- (6)}$$

is a 2×2 matrix.

Expanding (6) we get

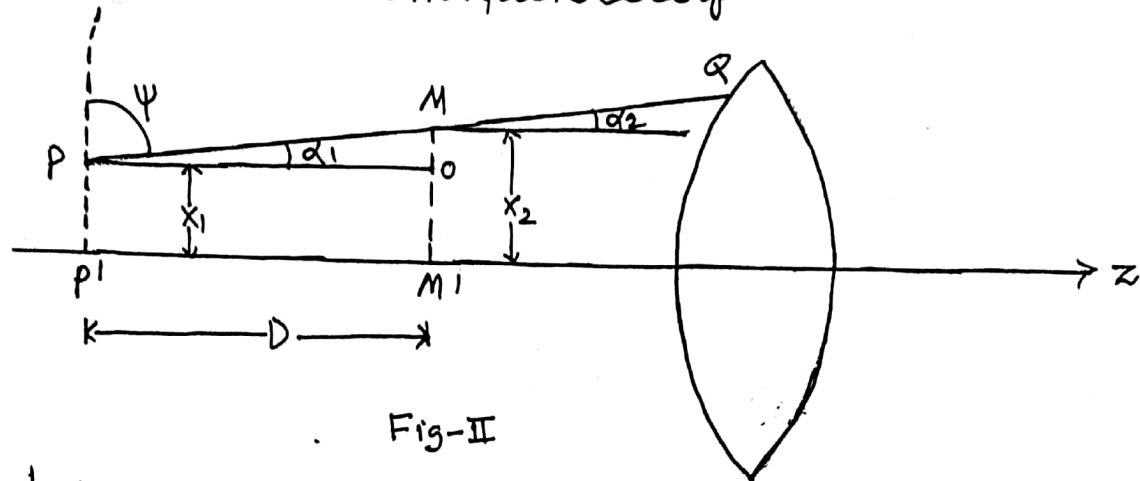
$$x_1 = (ae + bg)z_1 + (af + bh)z_2 \quad \text{--- (7a)}$$

$$x_2 = (ce + dg)z_1 + (cf + gh)z_2 \quad \text{--- (7b)}$$

The above formulation is widely used to solve linear equation by using matrix method.

In subsequent sections this method will be used to trace paraxial rays through a cylindrically symmetric optical system.

Matrix method for paraxial rays:



Let us consider a cylindrically symmetric optical system (symmetric about z axis) as shown in the diagram above. Obviously the axis of symmetry is the z axis. Further the method is applicable to paraxial rays only (since non paraxial rays lead to aberration). Paraxial rays are those rays which pass through the axis or very close to the axis and are specified by their distance from the axis. For the ray PQ , the point P is at a distance x_1 from axis and makes an angle α_1 with the axis. So the quantities (x_1, α_1) represent the co-ordinates of the ray. Instead of this the ray can also be represented by its optical direction cosine as follows.

If n - be the refractive index, ψ - angle between the ray and the z axis, then the product

$$\lambda = n \cos \psi \quad (8)$$

is known as the optical direction cosine. If α - be the angle made by the ray with the axis then

$$\lambda = n \sin \alpha \quad (\because \psi + \alpha = 90^\circ)$$

for small α , $\lambda \approx n \alpha \quad (9^*)$

When a ray propagates through an optical system, it undergoes two operations:

- i) translation
- ii) refraction.

The rays undergo translation when they propagate through a homogeneous medium as shown in the figure in the region PQ .

However when it strikes the interface of the two media, it undergoes refraction, as that which happens beyond Q .

We shall now study the effect of translation and refraction on the co-ordinates of the rays.

I. Effect of Translation

As shown in Fig-II, the ray PQ is assumed to travel in a homogeneous medium having refractive index n_1 . At posⁿ P, x_1 - distance from the axis and α_1 - angle with the axis. Hence the co-ordinates at P is (x_1, α_1) . Similarly the co-ordinates at M are (x_2, α_2) . Since the medium is homogeneous and the ray travels along the straight line PQ, we have

$$\alpha_1 = \alpha_2 \quad \text{--- (10)}$$

From the figure $x_2 = x_1 + OM$

$$\text{or } x_2 = x_1 + D \tan \alpha_1 \quad (P'M' = D) = PO \quad \text{--- (11)}$$

For paraxial rays the angles α 's are very small and hence we can have the app^x $\tan \alpha_1 \approx \alpha_1$. ($\alpha_1 \rightarrow$ being measured in radians)

With this eqⁿ (11) becomes

$$x_2 = x_1 + \alpha_1 D \quad \text{--- (12)}$$

From (9*)

$\rightarrow$ Define $\lambda_1 = n_1 \alpha_1$ and $\lambda_2 = n_2 \alpha_2$ then for the case of translation in the homogeneous medium we have

$n_1 = n_2$ and $\alpha_1 = \alpha_2$; which yields from above eqⁿ (13)

$$\lambda_2 = \lambda_1 \quad \text{--- (14)}$$

and

$$x_2 = x_1 + \frac{\lambda_1}{n_1} D = x_1 + \frac{D}{n_1} \lambda_1 \quad \text{--- (15)}$$

These two linear equations (14) and (15) can be combined by using the matrix method as:

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D/n_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad \text{--- (16)}$$

The above equation (16) is interpreted as follows:

"If a ray is initially specified by a (2×1) matrix having elements λ_1 and x_1 , then, the effect of propagation/translation through a distance D in a homogeneous medium of refractive index n_1 is given by a (2×2) matrix $T = \begin{pmatrix} 1 & 0 \\ D/n_1 & 1 \end{pmatrix}$ --- (17) and the final ray is given by eqⁿ (16)". The matrix (17) is called translation matrix and its determinant is

$$\det \begin{pmatrix} 1 & 0 \\ D/n_1 & 1 \end{pmatrix} = \begin{vmatrix} 1 & 0 \\ D/n_1 & 1 \end{vmatrix} = 1 \quad \text{--- (18)}$$

II Effect of Refraction:

We shall now determine the matrix, that would represent the effect of refraction of a ray of light from one medium to the other, separated by a spherical surface as shown in fig-III.

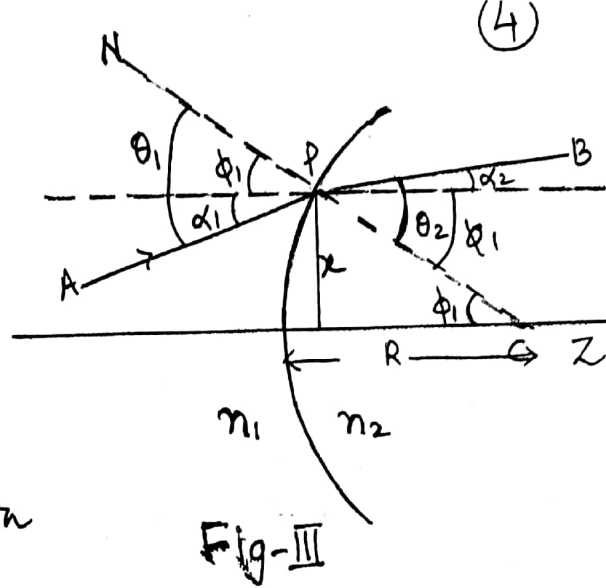


Fig-III

R - radius of curvature, ~~n1 and n2~~ n_1 and $n_2 \rightarrow$ r.i. of the two media

θ_1 and θ_2 - angles of incidence and angle of refraction.

AP - incident ray, PN - normal at point of incidence P, PB - refracted ray.

By Snell's Law $\frac{\sin \theta_1}{\sin \theta_2} = \frac{n_2}{n_1} \Rightarrow n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad (19)$

Since we are dealing with paraxial rays only, we have $\sin \theta_1 \approx \theta_1$ and $\sin \theta_2 \approx \theta_2$; such that (19) reduce to $n_1 \theta_1 = n_2 \theta_2 \quad (20)$

From the figure $\theta_1 = \phi_1 + \alpha_1$ and $\theta_2 = \phi_1 + \alpha_2$ | α_1 and α_2 are angle made by incident & refracted ray with Z axis

For paraxial rays ϕ_1 is small and hence

$$\tan \phi_1 = \frac{x}{R} \text{ or } \phi_1 \approx \frac{x}{R} \quad (22)$$

Using (21) and (22) in (20) we get

$$n_1 (\phi_1 + \alpha_1) \approx n_2 (\phi_1 + \alpha_2)$$

$$\Rightarrow n_2 \alpha_2 = n_1 \alpha_1 - (n_2 - n_1) \phi_1$$

$$\Rightarrow n_2 \alpha_2 = n_1 \alpha_1 - \frac{n_2 - n_1}{R} \cdot x \quad (23)$$

Using eq (23) $\lambda_2 = \lambda_1 - \frac{n_2 - n_1}{R} \cdot x$

$$\lambda_2 = \lambda_1 - P x \quad (24)$$

where $P = \frac{n_2 - n_1}{R} \quad (25)$

Further the height of the ray at P before and after refraction is same and let us write this as

$$x_2 = x_1 (= x) \quad (26a)$$

writing (24) in terms of x_1

$$\lambda_2 = \lambda_1 - P x_1 \quad (26b)$$

Called power of refracting surface.

These two equations (26) and (26) (a) can be expressed by the matrix formulation

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad \text{--- (27)}$$

This suggests that the refraction through a spherical surface can be characterised by a (2×2) matrix given by refraction matrix $R = \begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (n_2 - n_1)/R \\ 0 & 1 \end{pmatrix}$ --- (28)

Further it may be noted that, unlike the previous case

$$|R| = \det R = \begin{vmatrix} 1 & (n_2 - n_1)/R \\ 0 & 1 \end{vmatrix} = 1 \quad \text{--- (29)}$$

General formulation:

In general an optical system made up of series of lenses can be characterised by translation & refraction matrices.

If a ray entering an optical system is given by $\begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix}$ and the one leaving the system by $\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix}$, then eq (26) and (27) can be written in the general form

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} b & -a \\ -d & c \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad \text{--- (30)}$$

Where the matrix $S = \begin{pmatrix} b & -a \\ -d & c \end{pmatrix}$ is called System matrix and is solely determined by optical system. --- (31)

The -ve sign in some elements of S are chosen for convenience. We see that, a ray undergoes only two operations - translation and refraction, while traversing through an optical system, the system matrix in general is a product of translation & refraction matrices given by (17) and (28).

Further we note that $|T| = |R| = 1$ and $|TR| = |T||R|$.

$$\text{we get } |S| = 1. \quad \text{--- (32)}$$

$$\Rightarrow bc - ad = 1 \quad \text{--- (33)}$$

The quantities 'b' and 'c' being dimensionless, dim of 'a' and 'd' are inverse of one another.

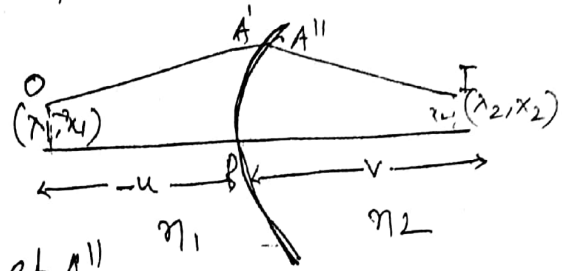
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Imaging by a Spherical Refracting Surface:

(6)

To illustrate the use of matrix formulation consider formation of image due to refraction through a spherical surface. The spherical surface separates two media of refractive index n_1 and n_2 .

Let the co-ordinates of the ray be (λ_1, x_1) at O; (λ', x') at A', (λ'', x'') at A'' and (λ_2, x_2) at I.



For the translation of the ray from O to A we have

$$\begin{pmatrix} \lambda' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D/n_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -u/n_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad \text{--- (34)} \quad \left| D = -u \text{ using the proper sign convention} \right|$$

For refraction of the ray from A' to A'', we have

$$\begin{pmatrix} \lambda'' \\ x'' \end{pmatrix} = \begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda' \\ x' \end{pmatrix} \quad \text{--- (35)}$$

and for translation from A'' to I, we have

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D/n_2 & 1 \end{pmatrix} \begin{pmatrix} \lambda'' \\ x'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v/n_2 & 1 \end{pmatrix} \begin{pmatrix} \lambda'' \\ x'' \end{pmatrix} \quad \text{--- (36)}$$

Using (35) and (36) in (34) we get

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v/n_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u/n_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad \text{--- (37)}$$

$$\Rightarrow \begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 + Pu/n_1 & -P \\ \frac{v}{n_2} \left(1 + Pu/n_1 \right) - \frac{u}{n_1} & \left(1 - \frac{vP}{n_2} \right) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad \text{--- (37)}$$

$$\Rightarrow x_2 = \left[\frac{v}{n_2} \left(1 + \frac{Pu}{n_1} \right) - \frac{u}{n_1} \right] \lambda_1 + \left(1 - \frac{vP}{n_2} \right) x_1 \quad \text{--- (37)*}$$

If the ray is emanating from an axial object point O i.e. $x_1 \rightarrow 0$, the image plane is determined by $x_2 = 0$. Using these two conditions in (37) we get

$$\frac{v}{n_2} \left(1 + Pu/n_1 \right) - \frac{u}{n_1} = 0 \quad \text{--- (38)}$$

$$\text{or } u/n_1 = \frac{v}{n_2} \left(1 + Pu/n_1 \right)$$

$$= \frac{n_2}{v} - \frac{n_1}{u} = P = \frac{n_2 - n_1}{R} \quad \text{--- (39) using eqn (25)}$$

Thus in this image plane we get

$$\begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 + P/n_1 & -P \\ 0 & 1 - vP/n_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} \quad \text{--- (40)}$$

obtained by using (38) in 37

This gives $x_2 = (1 - vP/n_2) x_1$

and the magnification is given by

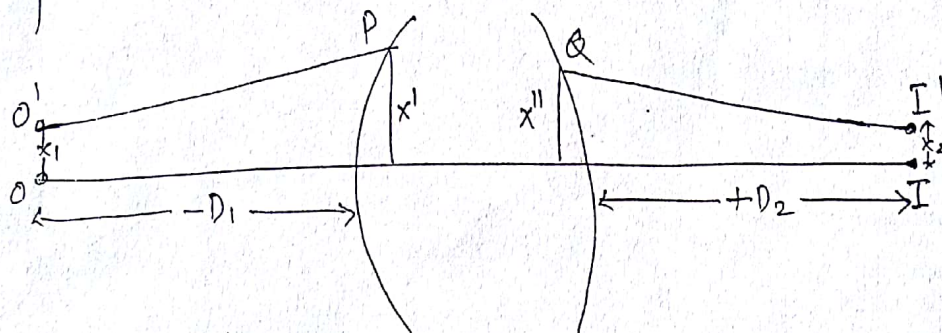
$$m = \frac{x_2}{x_1} = 1 - vP/n_2$$

$$= 1 - \frac{v}{n_2} \left(\frac{n_2}{v} - \frac{n_1}{u} \right) \quad \text{Using (39) for } P.$$

$$m = \frac{n_1 v}{n_2 u} \quad \text{--- (41)}$$

Formation of Image by a coaxial optical system:

As an illustration of the use of matrix method consider the formation of image by a medium bounded by two refracting surfaces.



As shown in figure above:

$OO' \rightarrow$ represent object plane

$II' \rightarrow$ image plane

$-D_1 \rightarrow$ Distance of 1st refracting surface from object plane (as per sign convention)

$+D_2 \rightarrow$ " 2nd refracting surface from image plane. ($\because$ Since D_2 is +ve, image is real).

$O'P \rightarrow$ represent the ray starting from O' which lies in object plane

$QI' \rightarrow$ " " " emerging from the 2nd refracting surface.

The point I' lie in the image plane. The point I is the paraxial image of object O

Let $(\lambda_1, x_1), (\lambda', x'), (\lambda'', x''), (\lambda_2, x_2) \rightarrow$ represent the Co-ordinates of the ray at O', P, Q, & I' respectively. (8)

Using the matrix formulation we get

$$\begin{pmatrix} \lambda' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -D_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix}; \begin{pmatrix} \lambda'' \\ x'' \end{pmatrix} = \begin{pmatrix} b & -a \\ -d & c \end{pmatrix} \begin{pmatrix} \lambda' \\ x' \end{pmatrix}; \text{ \& } \begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D_2 & 1 \end{pmatrix} \begin{pmatrix} \lambda'' \\ x'' \end{pmatrix}$$

which yields

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D_2 & 1 \end{pmatrix} \begin{pmatrix} b & -a \\ -d & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -D_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad (42)$$

(The refractive index for translation is 1, the medium being air).

$$\Rightarrow \begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} b + aD_1 & -a \\ bD_2 + aD_1D_2 - cD_1 - d & c - aD_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad (42)^*$$

$$\Rightarrow x_2 = (bD_2 + aD_1D_2 - cD_1 - d)\lambda_1 + (c - aD_2)x_1 \quad (43)$$

If the ray is emanating from axial object point i.e. $x_1 = 0$, the image plane is determined by $x_2 = 0$. With this the above eqⁿ becomes

$$bD_2 + aD_1D_2 - cD_1 - d = 0 \quad (44)$$

Using condⁿ (44) in (42)* we get

$$\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} b + aD_1 & -a \\ 0 & c - aD_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad (45)$$

If $x_2 \neq 0$, the above eqⁿ yields

$$x_2 = (c - aD_2)x_1 \text{ which yields the}$$

$$\text{magnification } m = \frac{x_2}{x_1} = (c - aD_2) \quad (46)$$

Further from (45) $\begin{vmatrix} b + aD_1 & -a \\ 0 & c - aD_2 \end{vmatrix} = 1$

$$\Rightarrow (b + aD_1)(c - aD_2) = 1$$

$$\Rightarrow b + aD_1 = \frac{1}{c - aD_2} = \frac{1}{m} \quad \text{Using (46)}$$

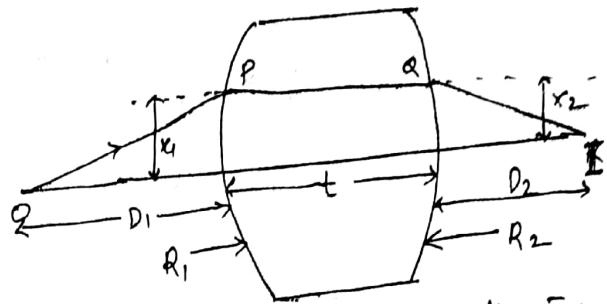
Thus if x_1 and x_2 correspond to object and image planes then for general optical system $\begin{pmatrix} \lambda_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/m & -a \\ 0 & m \end{pmatrix} \begin{pmatrix} \lambda_1 \\ x_1 \end{pmatrix} \quad (47)$ (using 47 in 45)

Example:

Obtain system matrix for a thick lens;
and hence derive the thin and thick lens formula.

Solution:

Consider a lens of thickness 't'
and made up of material of r.i. 'n'
 R_1 & R_2 - Radii of Curvature of the
two Surfaces.



A ray from O strikes 1st Surface
at P and emerge from the 2nd Surface
at Q.

A paraxial ray passing through
a thick lens of thickness 't'.

Let the Co-ordinates of the ray at P and Q are $\begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix}$ respectively.

Where λ_1 and λ_2 are the direction cosines at P & Q.

x_1 and x_2 are distance from the axis of symmetry.

The ray propagating from P $\rightarrow$ Q under goes two refractions
at P & Q and one translation through distance PQ $\approx t$ in the
medium of the lens having thickness refractive index 'n'.

We have now
$$\begin{pmatrix} x_2 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & -P_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t/n & 1 \end{pmatrix} \begin{pmatrix} 1 & -P_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} \quad (49)$$

Where $P_1 = \frac{n-1}{R_1}$, $P_2 = \frac{1-n}{R_2} = -\frac{n-1}{R_2}$ (51)

P_1 and P_2 represent the power of the two refracting surfaces.

Our system matrix becomes $S = \begin{pmatrix} b & -a \\ -d & c \end{pmatrix} = \begin{pmatrix} 1 & -P_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t/n & 1 \end{pmatrix} \begin{pmatrix} 1 & -P_1 \\ 0 & 1 \end{pmatrix}$

or
$$S = \begin{bmatrix} 1 - P_2 t/n & -P_1 - P_2(1 - P_1 t/n) \\ t/n & 1 - P_1 t/n \end{bmatrix} \quad (51)$$

For a thin lens $t \rightarrow 0 \Rightarrow S = \begin{bmatrix} 1 & -P_1 - P_2 \\ 0 & 1 \end{bmatrix} \quad (52)$

Comparing with (A) we get $a = P_1 + P_2$, $b = 1$, $c = 1$, $d = 0$ (53)

Substituting these values in eqn (44) (for axial object & image points)

i.e. namely $bD_2 + aD_1D_2 - cD_1 - d = 0$ becomes

$$D_2 + (P_1 + P_2)D_1D_2 - D_1 = 0$$

$$\Rightarrow \frac{1}{D_2} - \frac{1}{D_1} = P_1 + P_2 = \frac{n-1}{R_1} + \frac{n-1}{R_2} = (n-1) \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad (54)$$

$$\Rightarrow \boxed{\frac{1}{D_2} - \frac{1}{D_1} = \frac{1}{f}} \quad (55) \quad \text{Where } f = \frac{1}{P_1 + P_2} = \left[(n-1) \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right]^{-1} \quad (56)$$

is the famous lens formula, represents the focal length of the lens.

* for paraxial rays $PQ \approx t$, the thickness of the lens.

P.T.O

Eqⁿ (55) represent the lens formula.

The signs of R_1 and R_2 are to be properly chosen as shown in side figure

Fraunhofer's: from equation (51)

$$a = R_1 + R_2(1 - R_1 t/n)$$

$$b = 1 - R_2 t/n$$

$$c = 1 - R_1 t/n$$

$$d = -t/n$$

Substituting these values in eqⁿ (43) we can obtain the required relation between D_1 and D_2 .

